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MODEL COMPANIONS OF DISTRIBUTIVE p-ALGEBRAS

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§0. Introduction. Let $\underline{B}_n$, $0 \leq n \leq \omega$, be the equational classes of distributive p-algebras (precise definitions are given in §1). It has been known for some time that the elementary theories T_n of $\underline{B}_n$ possess model companions T_n^*; see, e.g., [6] and [14] and the references given there. However, no axiomatizations of T_n^* were given, with the exception of $n = 0$ (Boolean case) and $n = 1$ (Stonian case). While the first case belongs to the folklore of the subject (see [6], also [11]), the second case presented considerable difficulties (see Schmitt [13]). Schmitt's use of methods characteristic for Stone algebras seems to prevent a ready adaptation of his results to the cases $n \geq 2$.

The natural way to get a hold on T_n^* is to determine the class $E(\underline{B}_n)$ of existentially complete members of $\underline{B}_n$: Since T_n^* exists, it equals the elementary theory of $E(\underline{B}_n)$. The present author succeeded [12] in solving the simpler problem of determining the classes $A(\underline{B}_n)$ of algebraically closed algebras in $\underline{B}_n$ (exact definitions of $A(\underline{B}_n)$ and $E(\underline{B}_n)$ are given in §1) for all $0 \leq n \leq \omega$. $A(\underline{B}_n)$ is easier to handle since it contains sufficiently many "small" algebras–viz. finite direct products of certain subdirectly irreducibles–in terms of which the members of $A(\underline{B}_n)$ may be analyzed (in contrast, all members of $E(\underline{B}_n)$ are infinite and $\aleph_0$-homogeneous). As it turns out, $A(\underline{B}_n)$ is finitely axiomatizable for all n, and comparing the theories of $A(\underline{B}_0)$, $A(\underline{B}_1)$ with the explicitly known theories of $E(\underline{B}_0)$, $E(\underline{B}_1)$–viz. T_0^*, T_1^*–, a reasonable conjecture for T_n^*, $2 \leq n \leq \omega$, is immediate. The main part of this paper is concerned with verifying that the conditions formalized by T_n^* suffice to describe the algebras in $E(\underline{B}_n)$ (necessity is easy). This verification rests on the same combinatorial techniques as used in [12] to describe the members of $A(\underline{B}_n)$.

§1 gives the pertinent definitions. For anything not found there, the reader is referred to Grätzer [3, Chapter III, in particular] for the algebraic part and to Hirschfeld and Wheeler [6] for the model-theoretic side. In §2, we summarize the results on $A(\underline{B}_n)$ from [12] and characterize the members of $E(\underline{B}_n)$ within $A(\underline{B}_n)$ by four conditions, EC1 through EC4. Combination of these results yields the desired description of existentially complete algebras in $\underline{B}_n$ for $0 \leq n \leq \omega$. Formalizing these descriptions accordingly, T_n^* may be written down explicitly (§3) and shown to be $\aleph_0$-categorical and complete for all n, whereas only T_0^*, T_1^*, T_2^* and $T_\omega{}^*$ are even model completions of their respective T_i.

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§1. Definitions and notation. A (*distributive*) *p-algebra* L is an algebra $L(\wedge, \vee, {}^*, 0, 1)$ such that $L(\wedge, \vee, 0, 1)$ is a distributive lattice with universal bounds 0 and 1 and the unary operation * satisfies $x \leq a^*$ iff $x \wedge a = 0$. Since only distributive p-algebras are considered in this paper, "distributive" will be omitted in the sequel. The class of all p-algebras is equational and will be denoted by $\underline{B}_\omega$. The nontrivial equational subclasses of $\underline{B}_\omega$ may be enumerated in a sequence $\underline{B}_0 \subseteq \underline{B}_1 \subseteq \cdots \subseteq \underline{B}_n \subseteq \cdots \subseteq \underline{B}_\omega$ $(n \in \omega)$ (see Lee [10]). $\underline{B}_0$ is the class of all Boolean algebras, $\underline{B}_1$ that of all Stone algebras (those satisfying the identity $x^* \vee x^{**} = 1$). The easiest way to describe the classes $\underline{B}_n$ is by listing their subdirectly irreducible members. We need some notation. Let $\underline{2}$ be the two-element Boolean algebra, and put $F_n = \underline{2}_n$ for $n \in \omega$. For later reference, we agree to write $\underline{C}$ for the countable atomless Boolean algebra. If L is any lattice, $\hat{L}$ denotes the lattice obtained from L by adjoining a new greatest element to L. Now the subdirectly irreducible algebras in $\underline{B}_n$ $(n < \omega)$ are exactly $\hat{F}_0 \cong \underline{2}, \hat{F}_1, \ldots, \hat{F}_n$, while an algebra is subdirectly irreducible in $\underline{B}_\omega$ iff it is of the form $\hat{B}$ for some Boolean algebra B. For details, compare Chapter III of [3].

On the model-theoretic side, we use a first-order language $\mathscr{L}$ with equality. $\mathscr{L}$ has variables $x_1, x_2, \ldots$ and as nonlogical symbols two binary function symbols $\wedge, \vee$, a unary function symbol * and two constants 0, 1 with the obvious intended interpretations. We define $\mathscr{L}$-theories T_n for $0 \leq n \leq \omega$ as follows: T_ω consists of any convenient set of $\mathscr{L}$-sentences axiomatizing distributive lattices with 0, 1 together with $(\forall x_1, x_2)(x_1 \wedge (x_1 \wedge x_2)^* = x_1 \wedge x_2^*)$. For $n \geq 1$, let θ_n be the sentence

$$(\forall x_1, \ldots, x_n)((x_1 \wedge \cdots \wedge x_n)^* \vee (x_1^* \wedge x_2 \wedge \cdots \wedge x_n)^* \vee (x_1 \wedge x_2^* \wedge \cdots \wedge x_n)^* \vee \cdots \vee (x_1 \wedge x_2 \wedge \cdots \wedge x_n^*)^* = 1).$$

Let θ_0 be $(\forall x_1)(x_1 \vee x_1^* = 1)$. Now define $T_n = T_\omega \cup \{\theta_n\}$. Then $\underline{B}_n$ is exactly the class of models of T_n for $0 \leq n \leq \omega$ (see [10]).

Consider now any fixed $\underline{B}_n$, $0 \leq n \leq \omega$. $L \in \underline{B}_n$ is called *existentially complete* (abbreviated e.c.) iff for any $\exists_1$-sentence θ from $\mathscr{L}(L)$ and for any extension $L' \in \underline{B}_n$ of L, $L' \models \theta$ implies $L \models \theta$. L is called *algebraically closed* (abbreviated a.c.) if the same holds for *positive* $\exists_1$-sentences. We put $E(\underline{B}_n) = \{L \in \underline{B}_n;\ L \text{ is e.c.}\}$ and $A(\underline{B}_n) = \{L \in \underline{B}_n;\ L \text{ is a.c.}\}$. Hence $E(\underline{B}_n) \subseteq A(\underline{B}_n) \subseteq \underline{B}_n$ and all inclusions are strict, as we shall see. Given $L \in \underline{B}_\omega$, $SL = \{x \in L;\ x = x^{**}\}$ is the *skeleton* of L, $CL = \{x \in L;\ x \vee x^* = 1\}$ is the *center* of L and $DL = \{x \in L;\ x^* = 0\}$ is the filter of *dense* elements of L. Any $L \in \underline{B}_\omega$ contains a largest subalgebra which is *Stonian*, i.e., which belongs to $\underline{B}_1$. This is the subalgebra of L generated by $CL \cup DL$, and we denote it by Ston L. This definition is due to Katriňák; see, e.g., [7] for more details. Alternatively, Ston $L = \{x \in L;\ x^* \vee x^{**} = 1\}$. The following definition is adopted for technical convenience: Let $L \in \underline{B}_\omega$, $s \in$ Ston L. Define $B_L(s) = \{b \in SL;\ b \leq s \text{ and } b \vee b^* = s \vee s^*\} \cup \{0, s\}$. The subscript L will be omitted when there is no danger of confusion. In general, $B(s)$ is not closed under $\wedge$ or $\vee$; however $x \in B(s)$ implies $x^* \wedge s \in B(s)$ (since $x \vee (x^* \wedge s) = s$ and $x \wedge (x^* \wedge s) = 0$ this defines a relativized complementation on $B(s)$). Details may be found in [12]. Given $x \in L$ and a finite subset $\{y_1, \ldots, y_n\} \subseteq L$, we say that $\{y_1, \ldots, y_n\}$ is a *partition* of x provided $y_1 \vee \cdots \vee y_n = x$ and $y_i \wedge y_k = 0$ for $1 \leq i < k \leq n$. A partition is called *proper* iff it does not contain 0. Finally, given two p-algebras L_1, L_2, we allow ourselves to write $L_1 \subseteq L_2$ whenever L_2 contains an isomorphic copy of L_1 as a p-subalgebra.

§2. Existentially complete distributive p-algebras. The members of $A(\underline{B}_n)$ for $0 \leq n \leq \omega$ were determined in [12]. The following two theorems summarize the situation.

THEOREM 1 ($n < \omega$). *Let $L \in \underline{B}_n$, $0 \leq n < \omega$. The following are equivalent:*

(i) $L \in A(\underline{B}_n)$.

(ii) *L satisfies the following four conditions:*

AC1 *DL is relatively complemented.*

AC2 *For all d_1, $d_2 \in DL$ satisfying $d_1 \vee d_2 = 1$, there exists $c \in CL$ such that $c \leq d_1$, $c^* \leq d_2$.*

AC3n *Assume $n \geq 2$. For all $s \in$ Ston $L \backslash CL$, there exists a proper partition $\{b_1, \ldots, b_n\} \subseteq B_L(s)$ of s.*

AC4n *Assume $n \geq 2$. Put $N = 2^n + 1$. For all $s \in$ Ston $L \backslash CL$, every proper partition $\{b_1, \ldots, b_n\} \subseteq B_L(s)$ of s and every $0, s \neq b \in B_L(s)$, there exists a partition $\{c_1, \ldots, c_N\} \subseteq CL$ of $s^{**} \in CL$ such that $b = \bigvee\{(b_i \wedge c_j);\ 1 \leq i \leq n, 1 \leq j \leq N\}$.*

(iii) *If L_0 is a finite subalgebra of L, there exists a p-algebra L_1 such that $L_0 \subseteq L_1 \subseteq L$ and $L_1 \cong \underline{2}^i \times \hat{F}_n^j$ for some $i, j \in \omega$.*

THEOREM 1ω. *Let $L \in \underline{B}_\omega$. The following are equivalent:*

(i) $L \in A(\underline{B}_\omega)$.

(ii) *L satisfies the following four conditions:*

AC1 *as above.*

AC2 *as above.*

AC3ω *For all $s \in$ Ston $L \backslash CL$, $B_L(s) \supsetneqq \{0, s\}$.*

AC4ω *For all $s \in$ Ston $L \backslash CL$ and all $0, s \neq b \in B_L(s)$, there exists a proper partition $\{b_1, b_2\} \subseteq B_L(s)$ of b.*

(iii) *If L_0 is a finite subalgebra of L, there exists a p-algebra L_1 such that $L_0 \subseteq L_1 \subseteq L$ and $L_1 \cong \underline{2}^i \times \hat{\underline{C}}^j$ for some $i, j \in \omega$.*

The following definition lists the conditions necessary and sufficient to characterize the members of $E(\underline{B}_n)$ within $A(\underline{B}_n)$ for $0 \leq n \leq \omega$:

DEFINITION. Let $L \in \underline{B}_n$, $0 \leq n \leq \omega$. L will be said to satisfy.

EC1 iff CL has no atoms.

EC2 iff DL has no antiatoms;

EC3 iff for any $1 \neq c \in CL$ the set $\{d \in DL;\ 1 \geq d \geq c\}$ has no least element;

EC4 iff for any $0 \neq b \in SL$ there exists $0 \neq c \in CL$ such that $c \leq b$.

The following theorem contains the main result of this paper.

THEOREM 2. *An algebra is existentially complete in $\underline{B}_n$ ($0 \leq n \leq \omega$) iff it is algebraically closed and satisfies* EC1–EC4. *Alternatively, $L \in E(\underline{B}_n)$ iff L satisfies* AC1–AC4n *and* EC1–EC4.

Theorem 2 may be rephrased in a more compact form using Ston L and Theorem 3.2 of [13]:

COROLLARY 3. *$L \in E(\underline{B}_n)$ ($0 \leq n \leq \omega$) iff* Ston $L \in E(\underline{B}_1)$ *and L satisfies* AC3n, AC4n *and* EC4.

PROOF. Schmitt [13] proved that $L \in E(\underline{B}_1)$ iff L satisfies AC1, AC2 and EC1–EC3. ■

The proof of Theorem 2 will be broken down into a series of lemmata. We begin with the easy half of the theorem.

LEMMA 4. *Let* $L \in E(\underline{B}_n)$, $0 \le n \le \omega$. *Then* L *satisfies* EC1–EC4.

PROOF. Consider the following $\exists_1$-sentences from $\mathscr{L}(L)$:

$\theta_1(c)$: $(\exists x_1)(x_1 \vee x_1^* = 1 \;\&\; 0 < x_1 < c)$, $0 \ne c \in CL$,

$\theta_2(d)$: $(\exists x_1)(x_1^* = 0 \;\&\; d < x_1 < 1)$, $1 \ne d \in DL$,

$\theta_3(c, d)$: $(\exists x_1)(x_1^* = 0 \;\&\; c \le x_1 < d)$, $1 \ne c \in CL$ and $c < d \le 1$, $d \in DL$,

$\theta_4(b)$: $(\exists x_1)(x_1 \vee x_1^* = 1 \;\&\; x_1 \ne 0 \;\&\; x_1 \le b)$, $0 \ne b \in SL$.

Each one of these sentences may be satisfied in some direct product $L' \supseteq L$ of suitably many subdirectly irreducibles from $\underline{B}_n$, so they must hold in L. ■

LEMMA 5. *Let* $L \in \underline{B}_n$, $0 \le n \le \omega$, *and assume* $L \cong \prod(L_i, i \in I)$. *Then any of* EC1, ..., EC4 *holds in* L *iff it holds in every* L_i, $i \in I$.

PROOF. Straightforward. ■

We use the following notation: If $L \in \underline{B}_\omega$ and $a_1, \ldots, a_n \in L$, $\langle a_1, \ldots, a_n\rangle_L$ denotes the subalgebra of L generated by $\{a_1, \ldots, a_n\}$. A homomorphism f is *over* an algebra L iff $L \subseteq \operatorname{dom} f$ and f fixes L pointwise.

The following two lemmata take care of the essential cases of the sufficiency half of Theorem 2.

LEMMA 6. *Assume* $L \in \underline{B}_n$ $(2 \le n < \omega)$ *satisfies* AC1–AC4n *and* EC1–EC4. *Let* $L \subseteq L'$ *and* $L' \cong \hat{F}_n^T$ *(T some index set)*; $L_0 \subseteq L$ *and* $L_0 \cong F_r$ *for some* $1 \le r \le n$ *or* $L_0 \cong \hat{F}_n$; $L_1 \subseteq L'$ *and* $L_1 \cong F_s$ *for some* $1 \le s \le n$ *or* $L_1 \cong \hat{F}_n$. *Then there exists* $L_2 \subseteq L$, $L_2 \cong L_1$ *such that* $\langle L_0 \cup L_1\rangle$ *and* $\langle L_0 \cup L_2\rangle$ *are isomorphic over* L_0.

PROOF. Assume L, L', L_0 and L_1 are given as described. Let $p_1, \ldots, p_r$ be the atoms of L_0, $q_1, \ldots, q_s$ those of L_1; $d = p_1 \vee \cdots \vee P_r$, $\delta = q_1 \vee \cdots \vee q_s$; denote by S_n the group of permutations of an n-element set.

We define u_j $(1 \le j \le s)$, v_h $(h \in S_n)$, x_i $(1 \le i \le r)$ and y_{ij} $(1 \le i \le r,\ 1 \le j \le s)$ in CL' as follows by listing their components $(t \in T)$:

$$u_{jt} = \begin{cases} 1, & d_t \ne 1 \text{ and } q_{jt} = 1, \\ 0, & \text{otherwise} \end{cases} \qquad (L_0 \cong \hat{F}_n,\ r = n);$$

$$v_{ht} = \begin{cases} 1, & d_t = \delta_t \ne 1 \text{ and } q_{jt} = P_{h(j)t} \text{ for } 1 \le j \le s, \\ 0, & \text{otherwise} \end{cases} \qquad (L_0 \cong \hat{F}_n \cong L_1, r = s = n);$$

$$x_{it} = \begin{cases} 1, & \delta_t \ne 1 \text{ and } p_{it} = 1, \\ 0, & \text{otherwise} \end{cases} \qquad (L_1 \cong \hat{F}_n,\ s = n);$$

$$y_{ijt} = \begin{cases} 1, & p_{it} = 1 = q_{jt}, \\ 0, & \text{otherwise.} \end{cases}$$

It is fairly obvious that u_j, v_h, x_i, y_{ij} are central in L', pairwise disjoint and have join 1. Note that if $u_j \ne 0$, then $u_j \not\le d < 1$ and similarly for v_h, whereas x_i, $y_{ij} \le p_i \le d$. We will now use AC1 through EC4 to "simulate" these members of CL' within CL.

Suppose $d = 1$. Hence $p_i \in CL$ for $1 \le i \le r$. Use EC1 to find $\bar{x}_i$, $\bar{y}_{ij} \in CL$, pairwise disjoint, such that $p_i = \bar{x}_i \vee \bigvee_j \bar{y}_{ij}$ and $\bar{x}_i = 0$ iff $x_i = 0$, $\bar{y}_{ij} = 0$ iff $y_{ij} = 0$. Put $\bar{u}_j = \bar{v}_h = 0$ for $1 \le j \le s$, $h \in S_n$.

Suppose $d < 1$. Use EC4 to find $c_i \in CL$, $c_i \neq 0$, such that $c_i \leq p_i$ for $1 \leq i \leq r$. Use EC1 to find $\bar{x}_i$, $\bar{y}_{ij} \in CL$, pairwise disjoint, such that $c_i = \bar{x}_i \vee \bigvee_j \bar{y}_{ij}$ for $1 \leq i \leq r$, and $\bar{x}_i = 0$ iff $x_i = 0$, $\bar{y}_{ij} = 0$ iff $y_{ij} = 0$. Let $c = (c_1 \vee \cdots \vee c_r)^*$. It follows that $c \nleq d$. We want to construct $\bar{u}_j$, $\bar{v}_h \in CL$ for $1 \leq j \leq s$, $h \in S_n$, pairwise disjoint, $\nleq d$, $c = \bigvee_j \bar{u}_j \vee \bigvee_h \bar{v}_h$, $\bar{u}_j = 0$ iff $u_j = 0$, $\bar{v}_h = 0$ iff $v_h = 0$. It suffices to show that if $c \nleq d$, there exist nonzero $c_1, c_2 \in CL$ satisfying $c_1 \vee c_2 = c$, $c_1 \wedge c_2 = 0$, $c_1, c_2 \nleq d$. Now $c \nleq d$ implies $c^* \vee d < 1$. By EC3, we find $d_1 \in DL$ such that $c^* \vee d < d_1 < 1$. By AC1, there exists $d_2 \in DL$ such that $d_2 \wedge d_1 = c^* \vee d$, $d_2 \vee d_1 = 1$. By AC2, there exists $c_0 \in CL$ such that $c_0 \leq d_1$, $c_0^* \leq d_2$. It is easy to check that $c_1 = c \wedge c_0$, $c_2 = c \wedge c_0^*$ have the required properties.

L_2 will now be constructed by describing its atoms $Q_1, \ldots, Q_s$ "piecewise", that is, by listing the meets of $Q_1, \ldots, Q_s$ with $\bar{u}_j$, $\bar{v}_h$, $\bar{x}_i$, $\bar{y}_{ij}$. This is sufficient since all algebras of type F_r or $\hat{F}_r$ are generated, as p-algebras, by their atoms. Only $Q_j \wedge \bar{x}_i$ requires some preliminary work. Assume $\bar{x}_i \neq 0$. Apply EC2 to $\bar{x}_i^*$ in order to produce $d_i \in DL$ satisfying $\bar{x}_i \wedge d_i < \bar{x}_i$. Obviously, $\bar{x}_i \wedge d_i \in \mathrm{Ston}L \backslash CL$, so by AC3$n$ there exists a proper partition $\{\beta_{i1}, \ldots, \beta_{in}\} \subseteq B(\bar{x}_i \wedge d_i)$ of $\bar{x}_i \wedge d_i$ (as observed above, $\bar{x}_i \neq 0$ implies $L_1 \cong \hat{F}_n$ and thus $s = n$). Now put

$$Q_j = \bar{u}_j \vee \bigvee_{h \in S_n} (p_{h(j)} \wedge \bar{v}_h) \vee \bigvee_{\bar{x}_i \neq 0} \beta_{ij} \vee \bigvee_{i=1}^{r} (p_i \wedge \bar{y}_{ij})$$

and let $L_2 = \langle Q_1, \ldots, Q_s \rangle_L$. It is fairly obvious from the construction that $\langle L_0 \cup L_1 \rangle$ and $\langle L_0 \cup L_2 \rangle$ are isomorphic over L_0. ■

LEMMA 7. *Assume* $L \in \underline{B}_\omega$ *satisfies* AC1–AC4 *and* EC1–EC4. *Let* $L \subseteq L'$ *and* $L \cong \prod(\hat{A}_t, t \in T)$, *where* A_t *is an atomless Boolean algebra for each* $t \in T$; $L_0 \subseteq L$ *and* $L_0 \cong F_r$ *or* $L_0 \cong \hat{F}_r$ *for some* $r \in \omega$, $r \geq 1$; $L_1 \subseteq L'$ *and* $L_1 \cong F_s$ *or* $L_1 \cong \hat{F}_s$ *for some* $s \in \omega$, $s \geq 1$. *Then there exists* $L_2 \subseteq L$, $L_2 \cong L_1$, *such that* $\langle L_0 \cup L_1 \rangle$ *and* $\langle L_0 \cup L_2 \rangle$ *are isomorphic over* L_0.

PROOF. Assume L, L', L_0, L_1 are given as specified; let $p_1, \ldots, p_r$ be the atoms of L_0, $q_1, \ldots, q_s$ those of L_1; $d = p_1 \vee \cdots \vee p_r$, $\delta = q_1 \vee \cdots \vee q_s$. Let M be the set of all $r \times s$ (0, 1)-matrices having at least one 1 in each row and each column.

We define u_j $(1 \leq j \leq s)$, v_A $(A \in M)$, x_i $(1 \leq i \leq r)$ and y_{ij} $(1 \leq i \leq r,\ 1 \leq j \leq s)$ in CL' as follows by listing their components ($t \in T$):

$$u_{jt} = \begin{cases} 1, & d_t \neq 1 \text{ and } q_{jt} = 1, \\ 0, & \text{otherwise;} \end{cases}$$

$$v_{At} = \begin{cases} 1, & d_t = \delta_t \neq 1 \text{ and } p_{it} \wedge q_{jt} = 0 \text{ iff } \alpha_{ij} = 0, \text{ where } A = (\alpha_{ij}), \\ 0, & \text{otherwise;} \end{cases}$$

$$x_{it} = \begin{cases} 1, & \delta_t \neq 1 \text{ and } p_{it} = 1, \\ 0, & \text{otherwise;} \end{cases}$$

$$y_{ijt} = \begin{cases} 1, & p_{it} = 1 = q_{jt}, \\ 0, & \text{otherwise.} \end{cases}$$

Again, u_j, v_A, x_i, y_{ij} are central in CL', pairwise disjoint and have join 1; $u_j \neq 0$ implies $u_j \nleq d < 1$, and similarly for v_A.

Proceed as in the proof of Lemma 6 to obtain $\bar{u}_j$, $\bar{v}_A$, $\bar{x}_i$, $\bar{y}_{ij} \in CL$ which are 0 exactly if their counterparts in CL' are such, which are pairwise disjoint, have join 1, and satisfy $\bar{x}_i, \bar{y}_{ij} \leq p_i, \bar{u}_j, \bar{v}_A \not\leq d$ provided $d \leq 1$.

We construct $Q_1, \ldots, Q_s$ in the same way as in the proof of Lemma 6. Let $A = (\alpha_{ij}) \in M$. To obtain $Q_j \wedge \bar{v}_A$, consider $p_i \wedge \bar{v}_A$ for $i \leq 1 \leq r$: $p_i \wedge \bar{v}_A \in B(d \wedge \bar{v}_A)$, so use AC3$\omega$ and AC4ω to get a representation $p_i \wedge \bar{v}_A = \bigvee_{j=1}^{s} p_{ijA}$, where the p_{ijA} belong to $B(d_i \wedge \bar{v}_A)$, are disjoint and $p_{ijA} = 0$ iff $\alpha_{ij} = 0$. Put $Q_j \wedge \bar{v}_A = \bigvee_{i=1}^{r} p_{ijA}$. Next, assume $\bar{x}_i \neq 0$. Applying EC2 to $\bar{x}_1^*$ yields $d_i \in DL$ satisfying $\bar{x}_i \wedge d_i < \bar{x}_i$. By AC3$\omega$, $B(\bar{x}_i \wedge d_i) \neq \{0, \bar{x}_i \wedge d_i\}$, so using AC4$\omega$ suitably often one finds $\beta_{i1}, \ldots, \beta_{is} \in B(\bar{x}_i \wedge d_i)$, nonzero, pairwise disjoint and satisfying $\bar{x}_i \wedge d_i = \beta_{i1} \vee \cdots \vee \beta_{is}$. Put $Q_j \wedge \bar{x}_i = \beta_{ij}$. Finally, let

$$Q_j = \bar{u}_j \vee \bigvee_{A \in M} \bigvee_{i=1}^{r} p_{ijA} \vee \bigvee_{x_i \neq 0} \beta_{ij} \vee \bigvee_{i=1}^{r} (p_i \wedge \bar{y}_{ij}).$$

Define $L_2 = \langle Q_1, \ldots, Q_s \rangle_L$. By construction, $\langle L_0 \cup L_1 \rangle$ and $\langle L_0 \cup L_2 \rangle$ are isomorphic over L_0. ∎

Proof of Theorem 2, Sufficiency Part. Let $L \in \underline{B}_n$ $(2 \leq n \leq \omega)$ and assume L satisfies AC1–AC4n and EC1–EC4. Consider $L_1 \in \underline{B}_n$, $L_1 \supseteq L$; $a_1, \ldots, a_l \in L$, $v_1, \ldots, v_m \in L_1$. Proving $L \in E(\underline{B}_n)$ amounts to constructing $u_1, \ldots, u_m \in L$ such that $\langle a_1, \ldots, a_l\ v_1, \ldots, v_m \rangle$ and $\langle a_1, \ldots, a_l, u_1, \ldots, u_m \rangle$ are isomorphic over $\langle a_1, \ldots, a_l \rangle$: If $L_1 \models (\exists x_1, \ldots, x_m)\phi(x_1, \ldots, x_m, a_1, \ldots, a_l)$ with ϕ quantifier-free from $\mathscr{L}(L)$, say $L_1 \models \phi(v_1, \ldots, v_m, a_1, \ldots, a_l)$, then by isomorphism over $\langle a_1, \ldots, a_l \rangle$ we have $L \models \phi(u_1, \ldots, u_m, a_1, \ldots, a_l)$, that is, $L \models (\exists x_1, \ldots, x_m)\phi(x_1, \ldots, x_m, a_1, \ldots, a_l)$. Using subdirect representation and the fact that every Boolean algebra may be embedded into an atomless one, it clearly suffices to assume that

$$(*) \qquad L_1 \cong \hat{F}_n^T\ (n < \omega) \quad \text{or} \quad L_1 \cong \prod(\hat{A}_t, t \in T) \quad (n = \omega),$$

where A_t is an atomless Boolean algebra for each $t \in T$ (T any suitable index set).

Next, we may assume w.l.o.g. that $\{a_1, \ldots, a_l\}$ actually is a subalgebra of L. L is a.c. since it satisfies AC1–AC4n, so by Theorem 1(iii) there exists a subalgebra L' of L such that $\{a_1, \ldots, a_l\} \subseteq L'$ and $L' \cong \underline{2}^i \times \hat{F}_n^j\ (n < \omega)$ or $L' \cong \underline{2}^i \times \hat{C}^j$ $(n = \omega)$ for suitable $i, j \in \omega$. In the second case, we may conclude that $\{a_1, \ldots, a_l\} \subseteq \underline{2}^i \times \hat{F}_r^j$ for some $r \geq 1$. Hence, we may assume w.l.o.g. that $\{a_1, \ldots, a_l\}$ is isomorphic to $\underline{2}^i \times \hat{F}_n^j\ (n < \omega)$ or to $\underline{2}^i \times \hat{F}_r^j\ (n = \omega)$ for some i, j; r. The centers of these finite subalgebras of L contain $i + j$ atoms $c_1, \ldots, c_{i+j} \in CL \subseteq CL_1$. Divide L_1 by the canonical congruences $\theta(c_k)$, $1 \leq k \leq i + j$. $L_1/\theta(c_k)$ is still a direct product of type $(*)$, and $L/\theta(c_k)$ still satisfies EC1–EC4 by Lemma 5 and AC1–AC4n by Lemma 2.2 of [12]. Since $L \cong \prod(L/\theta(c_k), 1 \leq k \leq i + j)$, it will obviously suffice to construct the desired $u_1, \ldots, u_m$ modulo $\theta(c_k)$ for each k. Summing up, the problem reduces to the case where L_1 is a direct product of type $(*)$, and $\{a_1, \ldots, a_l\}$ is a subalgebra of L isomorphic to $\underline{2}$ or $\hat{F}_n\ (n < \omega)$ or to $\underline{2}$ or $\hat{F}_r$ for some $r \geq 1$ $(n = \omega)$.

We turn to $v_1, \ldots, v_m$. Observe L_1 is a.c. as a direct product of a.c. factors [12, Lemma 2.2]. So we may proceed as above and replace $\{v_1, \ldots, v_m\}$ by a subalgebra of L_1 isomorphic to $\underline{2}^p \times \hat{F}_n^q\ (n < \omega)$ or to $\underline{2}^p \times \hat{F}_r^q\ (n = \omega)$ for suitable p, q; r. The centers of these finite algebras contain $p + q$ atoms $c_1', \ldots, c_{p+q}' \in CL_1$.

To carry out the same factorization as above, we have to find $c_1, \ldots, c_{p+q} \in CL$ (nonzero, pairwise disjoint, with join 1) and $v'_1, \ldots, v'_m \in L_1$ such that

$$\begin{aligned} j_k(a_1/\theta(c'_k)) &= a_1/\theta(c_k) \\ &\vdots \\ j_k(a_l/\theta(c'_k)) &= a_l/\theta(c_k) \\ j_k(v_1/\theta(c'_k)) &= v'_1/\theta(c_k) \\ &\vdots \\ j_k(v_m/\theta(c'_k)) &= v'_m/\theta(c_k) \end{aligned}$$

induces an isomorphism

$$j_k\colon \langle a_1, \ldots, a_l, \ v_1, \ldots, v_m\rangle/\theta(c'_k) \cong \langle a_1, \ldots, a_l, \ v'_1, \ldots, v'_m\rangle/\theta(c_k).$$

If $\{a_1, \ldots, a_l\}$ is $\underline{2}$, let $\{c_1, \ldots, c_{p+q}\}$ be an arbitrary proper central partition of 1 in L. If $\{a_1, \ldots, a_l\}$ is $\hat{F}_r$ for some $r \in \omega$, let $\beta_1, \ldots, \beta_r$ be the atoms of $\hat{F}_r$ and d their join. Proceed as in the proof of Lemma 6 to produce $c_k \in CL$, $1 \leq k \leq p + q$ (nonzero, disjoint, with join 1) such that $c_k \leq d$ iff $c'_k \leq d$ and $c_k \wedge \beta_i \neq 0$ iff $c'_k \wedge \beta_i \neq 0$ for $1 \leq i \leq r$. Obviously, then, j_k as defined above will induce an isomorphism $\langle a_1, \ldots, a_l\rangle/\theta(c'_k) \cong \langle a_1, \ldots. a_l\rangle/\theta(c_k)$. $v'_1, \ldots, v'_m \in L_1$ will be constructed in the same way as $Q_1, \ldots, Q_s$ were obtained in the proof of Lemma 7; the difference being that all the auxiliary elements used in that construction live trivially within the direct product L_1 so we need not appeal to the EC and AC conditions at this point (except for EC1 which guarantees the existence of arbitrarily fine central partitions of c_k within L, thus within L_1).

Now the problem of finding the required $u_1, \ldots, u_m \in L$ may be factorized again since $L \cong \prod(L/\theta(c_k)$, $1 \leq k \leq p + q)$. Observing that every nontrivial homomorphic image of $\hat{F}_r$ $(r \in \omega)$ is some F_s $(s \leq r)$, we are reduced to considering the two cases dealt with in Lemmata 6 and 7. In view of Lemma 4, the proof of Theorem 2 is now complete, since the cases $\underline{B}_0$, $\underline{B}_1$ are known [11], [13]. ■

§3. Model companions for T_n. The existence of T_n^*, the model companion of T_n, for $0 \leq n \leq \omega$ has been known for some time. As far as the author knows, it appeared in print first in Burris [1]. However, as noted there, no description of the theories T_n^* was known then. T_0^* belongs to the folklore of the subject: It is the elementary theory of atomless Boolean algebras, see, e.g., [6] or, for an elementary account, [11]. An explicit description of T_1^* appeared in Schmitt [13]. Schmitt's constructions are based on some specific features of Stone algebras: The availability of a workable "triple" characterization of Stone algebras, and the coincidence between skeleton and center in such algebras. While the second property fails for $n \geq 2$, triple constructions for algebras in $\underline{B}_n$ exist for $n \geq 2$; see Katriňák [8] and [9]. Their technical complexity seems, however, to prevent a ready adaptation of Schmitt's techniques to the cases $n \geq 2$. A further existence proof for T_0^*, T_1^* and T_2^* was given by Weispfenning in [14]. No axiomatization of T_2^* is provided there, however, and the absence of the amalgamation property in $\underline{B}_n$ for $2 < n < \omega$ prevents a direct extension of Weispfenning's results to T_n for these values of n.

It is well known that if $E(\underline{K})$ is a generalized elementary class for $\underline{K}$ the class of all models of some universal theory T, then $T^* = \mathrm{Th}(E(\underline{K}))$ is the model companion of T. The T_n are obviously universal for $0 \leq n \leq \omega$. Now, for $L \in \underline{B}_n$, $0 \leq n \leq \omega$, the sets CL, SL, DL, Ston L, $B_L(s)$ are clearly definable by formulae from $\mathscr{L}$ ($\mathscr{L}(L)$ for the last); and so is the concept of a partition (proper partition) of fixed length. It follows that conditions AC1–AC4n and EC1–EC4 may be formalized by $\forall_2$-sentences from $\mathscr{L}$. Let ϕ_1, ϕ_2, $\phi_3(n)$, $\phi_4(n)$ be such formalizations of AC1, AC2, AC3n, AC4n for $2 \leq n \leq \omega$, and similarly θ_i for ECi ($1 \leq i \leq 4$). We may now rephrase Theorem 2 as follows:

THEOREM 8. *The model companions* T_n^* *of* T_n *for* $0 \leq n \leq \omega$ *are given by*:

$T_0^* = T_0 \cup \{\phi_1\}$,

$T_1^* = T_1 \cup \{\phi_1, \phi_2, \theta_1, \theta_2, \theta_3\}$,

$T_n^* = T_n \cup \{\phi_1, \phi_2, \phi_3(n), \phi_4(n), \theta_1, \theta_2, \theta_3, \theta_4\}$ *for* $2 \leq n \leq \omega$.

Hence, T_n^* *is finitely axiomatizable for all n.*

COROLLARY 9. T_n^* *is* $\aleph_0$*-categorical for all n.*

PROOF. See Burris [1]. ■

COROLLARY 10. T_n^* *is a model completion of* T_n *precisely for* $n = 0, 1, 2, \omega$.

PROOF. $\underline{B}_n = \mathrm{Mod}(T_n)$ has the amalgamation property exactly for these values of n (see [4]). The result follows (see [2]). ■

COROLLARY 11. T_n^* *is a complete theory for all n.*

PROOF. T_n^* is complete iff $\underline{B}_n = \mathrm{Mod}(T_n)$ has the joint embedding property (see [6]). Now $\underline{2}$ is an absolute subretract in $\underline{B}_n$ for all n (see [5]), hence $L_1, L_2 \in \underline{B}_n$ may be embedded into $L_1 \times L_2 \in \underline{B}_n$. ■

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